

# Characterizations of Fuzzy Fated Filters of $R_0$ -algebras Based on Fuzzy Points

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**Abstract.** More general form of the notion of quasi-coincidence of a fuzzy point with a fuzzy subset is considered, and generalization of fuzzy fated of  $R_0$ -algebras is discussed. The notion of an  $(\in, \in \vee q_k)$ -fuzzy fated filter in a  $R_0$ -algebra is introduced, and several properties are investigated. Characterizations of an  $(\in, \in \vee q_k)$ -fuzzy fated filter in an  $R_0$ -algebra are discussed. Using a collection of fated filters, a  $(\in, \in \vee q_k)$ -fuzzy fated filter is established.

**Keywords:** (Fated) filter, Fuzzy (fated) filter,  $(\in, \in \vee q)$ -fuzzy (fated) filter,  $(\in, \in \vee q_k)$ -fuzzy fated filter, Strong  $(\in, \in \vee q_k)$ -fuzzy fated filter.

## 1. Introduction

One important task of artificial intelligence is to make the computers simulate beings in dealing with certainty and uncertainty in information. Logic appears in a “sacred” (respectively, a “profane”) form which is dominant in proof theory (respectively, model theory). The role of logic in mathematics and computer science is twofold – as a tool for applications in both areas, and a technique for laying the foundations. Non-classical logic including many-valued logic, fuzzy logic, etc., takes the advantage of classical logic to handle information with various facets of uncertainty (see [14] for generalized theory of uncertainty), such as fuzziness, randomness etc. Non-classical logic has become a formal and useful tool for computer science to deal with fuzzy information and uncertain information. Among all kinds of uncertainties, incomparability is an important one which can be encountered in our life. The concept of  $R_0$ -algebras was first introduced by Wang in [11] by providing an algebraic proof of the completeness

theorem of a formal deductive system [12]. Obviously,  $R_0$ -algebras are different from the BL-algebras. Further, Pei and Wang [8] proved  $NM$ -algebras are categorically isomorphic to  $R_0$ -algebras. Jun and Liu [4] studied (fated) filters of  $R_0$ -algebras. They mentioned that the theory of  $R_0$ -algebras becomes one of the theoretical applications to the development of the theory of  $MTL$ -algebras. Some concrete practical and theoretical applications of  $R_0$ -algebras can be found in [8, 11]. Pei [9] proposed a new kind of fuzzy algebraic structure with the purpose to extending the concept of  $R_0$ -algebras and BL-algebras using results of normal residuated lattices into fuzzy settings. Liu and Li [5] discussed the fuzzy set theory of filters in  $R_0$ -algebras.

The algebraic structure of set theories dealing with uncertainties has been studied by some authors. The most appropriate theory for dealing with uncertainties is the theory of fuzzy sets developed by Zadeh [13]. Murali [7] proposed a definition of a fuzzy point belonging to fuzzy subset under a natural equivalence on fuzzy subset. The idea of quasi-coincidence of a fuzzy point with a fuzzy set, which is mentioned in [10], played a vital role to generate some different types of fuzzy subsets. It is worth pointing out that Bhakat and Das [1, 2] initiated the concepts of  $(\alpha, \beta)$ -fuzzy subgroups by using the “belongs to” relation  $(\in)$  and “quasi-coincident with” relation  $(q)$  between a fuzzy point and a fuzzy subgroup, and introduced the concept of an  $(\in, \in \vee q)$ -fuzzy subgroup. In particular, an  $(\in, \in \vee q)$ -fuzzy subgroup is an important and useful generalization of Rosenfeld’s fuzzy subgroup. As a generalization of the notion of fuzzy filters in  $R_0$ -algebras, Ma et al. [6] dealt with the notion of  $(\in, \in \vee q)$ -fuzzy filters in  $R_0$ -algebras. In [3], Han et al. dealt with the fuzzy set theory of fated filters in  $R_0$ -algebras. They provided conditions for a fuzzy filter to be a fuzzy fated filter, and introduced the notion of  $(\in, \in \vee q)$ -fuzzy fated filters. They established a relation between an  $(\in, \in \vee q)$ -fuzzy filter and an  $(\in, \in \vee q)$ -fuzzy fated filter, and provided conditions for an  $(\in, \in \vee q)$ -fuzzy filter to be an  $(\in, \in \vee q)$ -fuzzy fated filter. They also dealt with characterizations of an  $(\in, \in \vee q)$ -fuzzy fated filter. It is now natural to investigate more general type of  $(\in, \in \vee q)$ -fuzzy fated filters of an  $R_0$ -algebra. As a first step in this direction, we introduce the concept of an  $(\in, \in \vee q_k)$ -fuzzy fated filter of an  $R_0$ -algebra, and discuss some fundamental aspects of  $(\in, \in \vee q_k)$ -fuzzy fated filters. We deal with characterizations of  $(\in, \in \vee q_k)$ -fuzzy fated filters. Using a collection of fated filters, we establish an  $(\in, \in \vee q_k)$ -fuzzy fated filter.

The important achievement of the study with an  $(\in, \in \vee q_k)$ -fuzzy fated filter is that the notion of an  $(\in, \in \vee q)$ -fuzzy fated filter is a special case of an  $(\in, \in \vee q_k)$ -fuzzy fated filter, and thus so many results in the paper [3] are corollaries of our results obtained in this paper.

## 2. Preliminaries

Let  $L$  be a bounded distributive lattice with order-reversing involution  $\neg$  and a binary operation  $\rightarrow$ . Then  $(L, \wedge, \vee, \neg, \rightarrow)$  is called an  $R_0$ -algebra (see [11]) if it

satisfies the following axioms:

- (R1)  $x \rightarrow y = \neg y \rightarrow \neg x$ ,
- (R2)  $1 \rightarrow x = x$ ,
- (R3)  $(y \rightarrow z) \wedge ((x \rightarrow y) \rightarrow (x \rightarrow z)) = y \rightarrow z$ ,
- (R4)  $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$ ,
- (R5)  $x \rightarrow (y \vee z) = (x \rightarrow y) \vee (x \rightarrow z)$ ,
- (R6)  $(x \rightarrow y) \vee ((x \rightarrow y) \rightarrow (\neg x \vee y)) = 1$ .

Let  $L$  be an  $R_0$ -algebra. For any  $x, y \in L$ , we define  $x \odot y = \neg(x \rightarrow \neg y)$  and  $x \oplus y = \neg x \rightarrow y$ . It is proved that  $\odot$  and  $\oplus$  are commutative, associative and  $x \oplus y = \neg(\neg x \odot \neg y)$ , and  $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$  is a residuated lattice.

For any elements  $x, y$  and  $z$  of an  $R_0$ -algebra  $L$ , we have the following properties (see [8]).

- (a1)  $x \leq y$  if and only if  $x \rightarrow y = 1$ ,
- (a2)  $x \leq y \rightarrow x$ ,
- (a3)  $\neg x = x \rightarrow 0$ ,
- (a4)  $(x \rightarrow y) \vee (y \rightarrow x) = 1$ ,
- (a5)  $x \leq y$  implies  $y \rightarrow z \leq x \rightarrow z$ ,
- (a6)  $x \leq y$  implies  $z \rightarrow x \leq z \rightarrow y$ ,
- (a7)  $((x \rightarrow y) \rightarrow y) \rightarrow y = x \rightarrow y$ ,
- (a8)  $x \vee y = ((x \rightarrow y) \rightarrow y) \wedge ((y \rightarrow x) \rightarrow x)$ ,
- (a9)  $x \odot \neg x = 0$  and  $x \oplus \neg x = 1$ ,
- (a10)  $x \odot y \leq x \wedge y$  and  $x \odot (x \rightarrow y) \leq x \wedge y$ ,
- (a11)  $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z)$ ,
- (a12)  $x \leq y \rightarrow (x \odot y)$ ,
- (a13)  $x \odot y \leq z$  if and only if  $x \leq y \rightarrow z$ ,
- (a14)  $x \leq y$  implies  $x \odot z \leq y \odot z$ ,
- (a15)  $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$ ,
- (a16)  $(x \rightarrow y) \odot (y \rightarrow z) \leq x \rightarrow z$ .

A non-empty subset  $A$  of an  $R_0$ -algebra  $L$  is called a *filter* of  $L$  if it satisfies the following two conditions:

- (b1)  $1 \in A$ .
- (b2)  $(\forall x \in A) (\forall y \in L) (x \rightarrow y \in A \implies y \in A)$ .

It can be easily verified that a non-empty subset  $A$  of an  $R_0$ -algebra  $L$  is a filter of  $L$  if and only if it satisfies the following conditions:

- (b3)  $(\forall x, y \in A) (x \odot y \in A)$ .
- (b4)  $(\forall y \in L) (\forall x \in A) (x \leq y \implies y \in A)$ .

**Definition 2.1.** A fuzzy subset  $\mu$  of an  $R_0$ -algebra  $L$  is called a *fuzzy filter* of  $L$  if it satisfies:

- (c1)  $(\forall x, y \in L) (\mu(x \odot y) \geq \min\{\mu(x), \mu(y)\})$ .

(c2)  $\mu$  is order-preserving, that is,  $(\forall x, y \in L) (x \leq y \implies \mu(x) \leq \mu(y))$ .

**Theorem 2.2.** *A fuzzy subset  $\mu$  of an  $R_0$ -algebra  $L$  is a fuzzy filter of  $L$  if and only if it satisfies:*

(c3)  $(\forall x \in L) (\mu(1) \geq \mu(x))$ ,

(c4)  $(\forall x, y \in L) (\mu(y) \geq \min\{\mu(x \rightarrow y), \mu(x)\})$ .

For any fuzzy subset  $\mu$  of  $L$  and  $t \in (0, 1]$ , the set

$$U(\mu; t) = \{x \in L \mid \mu(x) \geq t\}$$

is called a *level subset* of  $L$ . It is well known that a fuzzy subset  $\mu$  of  $L$  is a fuzzy filter of  $L$  if and only if the non-empty level subset  $U(\mu; t)$ ,  $t \in (0, 1]$ , of  $\mu$  is a filter of  $L$ .

A fuzzy subset  $\mu$  of a set  $L$  of the form

$$\mu(y) := \begin{cases} t \in (0, 1] & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases}$$

is said to be a *fuzzy point* with support  $x$  and value  $t$  and is denoted by  $x_t$ .

For a fuzzy point  $x_t$  and a fuzzy subset  $\mu$  of  $L$ , Pu and Liu [10] introduced the symbol  $x_t \alpha \mu$ , where  $\alpha \in \{\in, q, \in \vee q\}$ . We say that

- (i)  $x_t$  belong to  $\mu$ , denoted by  $x_t \in \mu$ , if  $\mu(x) \geq t$ ,
- (ii)  $x_t$  is quasi-coincident with  $\mu$ , denoted by  $x_t q \mu$ , if  $\mu(x) + t > 1$ ,
- (iii)  $x_t \in \vee q \mu$  if  $x_t \in \mu$  or  $x_t q \mu$ ,

### 3. Generalizations of fuzzy fated filters based on fuzzy points

In what follows,  $L$  is an  $R_0$ -algebra unless otherwise specified. In [4], the notion of a fated filter of  $L$  is introduced as follows.

A non-empty subset  $A$  of  $L$  is called a *fated filter* of  $L$  (see [4]) if it satisfies (b1) and

$$(\forall x, y \in L) (\forall a \in A) (a \rightarrow ((x \rightarrow y) \rightarrow x) \in A \implies x \in A). \quad (1)$$

**Lemma 3.1.** *[[4]] A filter  $F$  of  $L$  is fated if and only if the following assertion is valid.*

$$(\forall x, y, z \in L) (x \rightarrow (y \rightarrow z) \in F, x \rightarrow y \in F \implies x \rightarrow z \in F). \quad (2)$$

**Lemma 3.2.** *[[4]] A filter  $F$  of  $L$  is fated if and only if the following assertion is valid.*

$$(\forall x, y \in L) ((x \rightarrow y) \rightarrow x \in F \implies x \in F). \quad (3)$$

Denote by  $FF(L)$  the set of all fated filters of  $L$ . Note that  $FF(L)$  is a complete lattice under the set inclusion with the largest element  $L$  and the least element  $\{1\}$ .

In what follows, let  $k$  denote an arbitrary element of  $[0, 1)$  unless otherwise specified. To say that  $x_t \in q_k \mu$ , we mean  $\mu(x) + t + k > 1$ . To say that  $x_t \in \vee q_k \mu$ , we mean  $x_t \in \mu$  or  $x_t \in q_k \mu$ .

**Definition 3.3.** A fuzzy subset  $\mu$  of  $L$  is said to be an  $(\in, \in \vee q_k)$ -fuzzy fated filter of  $L$  if it satisfies:

$$(1) \quad x_t \in \mu \implies 1_t \in \vee q_k \mu,$$

$$(2) \quad (a \rightarrow ((x \rightarrow y) \rightarrow x))_t \in \mu, \quad a_s \in \mu \implies x_{\min\{t,s\}} \in \vee q_k \mu$$

for all  $x, a, y \in L$  and  $t, s \in (0, 1]$ .

If a fuzzy subset  $\mu$  of  $L$  satisfies (c3) and Definition 3.3(2), then we say that  $\mu$  is a *strong*  $(\in, \in \vee q_k)$ -fuzzy fated filter of  $L$ . A (strong)  $(\in, \in \vee q_k)$ -fuzzy fated filter of  $L$  with  $k = 0$  is called a (strong)  $(\in, \in \vee q)$ -fuzzy fated filter of  $L$ .

*Example 3.4.* Let  $L = \{0, a, b, c, d, 1\}$  be a set with the following Hasse diagram and Cayley tables:

<div><div>1</div><div></div><div>d</div><div></div><div>c</div><div></div><div>b</div><div></div><div>c</div><div></div><div>d</div><div></div><div>a</div><div></div><div>0</div></div>	<table><tr><th><math>x</math></th><th><math>\neg x</math></th></tr><tr><td>0</td><td>1</td></tr><tr><td><math>a</math></td><td><math>d</math></td></tr><tr><td><math>b</math></td><td><math>c</math></td></tr><tr><td><math>c</math></td><td><math>b</math></td></tr><tr><td><math>d</math></td><td><math>a</math></td></tr><tr><td>1</td><td>0</td></tr></table>	$x$	$\neg x$	0	1	$a$	$d$	$b$	$c$	$c$	$b$	$d$	$a$	1	0	<table><tr><th><math>\rightarrow</math></th><th>0</th><th><math>a</math></th><th><math>b</math></th><th><math>c</math></th><th><math>d</math></th><th>1</th></tr><tr><td>0</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td></tr><tr><td><math>a</math></td><td><math>d</math></td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td></tr><tr><td><math>b</math></td><td><math>c</math></td><td><math>c</math></td><td>1</td><td>1</td><td>1</td><td>1</td></tr><tr><td><math>c</math></td><td><math>b</math></td><td><math>b</math></td><td><math>b</math></td><td>1</td><td>1</td><td>1</td></tr><tr><td><math>d</math></td><td><math>a</math></td><td><math>a</math></td><td><math>b</math></td><td><math>c</math></td><td>1</td><td>1</td></tr><tr><td>1</td><td>0</td><td><math>a</math></td><td><math>b</math></td><td><math>c</math></td><td><math>d</math></td><td>1</td></tr></table>	$\rightarrow$	0	$a$	$b$	$c$	$d$	1	0	1	1	1	1	1	1	$a$	$d$	1	1	1	1	1	$b$	$c$	$c$	1	1	1	1	$c$	$b$	$b$	$b$	1	1	1	$d$	$a$	$a$	$b$	$c$	1	1	1	0	$a$	$b$	$c$	$d$	1
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Then  $(L, \wedge, \vee, \neg, \rightarrow, 0, 1)$  is an  $R_0$ -algebra (see [5]), where  $x \wedge y = \min\{x, y\}$  and  $x \vee y = \max\{x, y\}$ . Define a fuzzy subset  $\mu$  of  $L$  by

$$\mu : L \rightarrow [0, 1], \quad x \mapsto \begin{cases} 0.7 & \text{if } x = 1, \\ 0.6 & \text{if } x = c, \\ 0.4 & \text{if } x = d, \\ 0.2 & \text{if } x \in \{0, a, b\}. \end{cases}$$

It is routine to verify that  $\mu$  is a strong  $(\in, \in \vee q_{0.4})$ -fuzzy fated filter of  $L$ . A fuzzy subset  $\nu$  of  $L$  defined by

$$\nu : L \rightarrow [0, 1], \quad x \mapsto \begin{cases} 0.8 & \text{if } x \in \{c, d\}, \\ 0.7 & \text{if } x = 1, \\ 0.3 & \text{if } x \in \{0, a, b\}. \end{cases}$$

is an  $(\in, \in \vee q_{0.2})$ -fuzzy fated filter of  $L$ , but it is not a strong  $(\in, \in \vee q_{0.2})$ -fuzzy fated filter of  $L$ .

**Theorem 3.5.** *Every  $(\in, \in)$ -fuzzy fated filter of  $L$  is an  $(\in, \in \vee q_k)$ -fuzzy fated filter.*

*Proof.* Straightforward. ■

Taking  $k = 0$  in Theorem 3.5, we have the following corollary.

**Corollary 3.6.** *Every  $(\in, \in)$ -fuzzy fated filter of  $L$  is an  $(\in, \in \vee q)$ -fuzzy fated filter.*

The converse of Theorem 3.5 is not true as seen in the following example.

*Example 3.7.* The  $(\in, \in \vee q_{0.2})$ -fuzzy fated filter  $\nu$  of  $L$  in Example 3.4 is not an  $(\in, \in)$ -fuzzy fated filter of  $L$ .

Obviously, every strong  $(\in, \in \vee q_k)$ -fuzzy fated filter is an  $(\in, \in \vee q_k)$ -fuzzy fated filter, but not converse as seen in Example 3.4.

We provided characterizations of an  $(\in, \in \vee q_k)$ -fuzzy fated filter.

**Theorem 3.8.** *A fuzzy subset  $\mu$  of  $L$  is an  $(\in, \in \vee q_k)$ -fuzzy fated filter of  $L$  if and only if it satisfies the following inequalities:*

- (1)  $(\forall x \in L) (\mu(1) \geq \min\{\mu(x), \frac{1-k}{2}\})$ ,
- (2)  $(\forall x, a, y \in L) (\mu(x) \geq \min\{\mu(a \rightarrow ((x \rightarrow y) \rightarrow x)), \mu(a), \frac{1-k}{2}\})$ .

*Proof.* Let  $\mu$  be an  $(\in, \in \vee q_k)$ -fuzzy fated filter of  $L$ . Assume that there exists  $a \in L$  such that  $\mu(1) < \min\{\mu(a), \frac{1-k}{2}\}$ . Then  $\mu(1) < t \leq \min\{\mu(a), \frac{1-k}{2}\}$  for some  $t \in (0, \frac{1-k}{2}]$ , and so  $a_t \in \mu$ . It follows from Definition 3.3(1) that  $1_t \in \vee q_k \mu$ , i.e.,  $1_t \in \mu$  or  $1_t q_k \mu$  so that  $\mu(1) \geq t$  or  $\mu(1) + t + k > 1$ . This is a contradiction. Hence  $\mu(1) \geq \min\{\mu(x), \frac{1-k}{2}\}$  for all  $x \in L$ . Suppose that there exist  $a, b, c \in L$  such that

$$\mu(b) < \min\{\mu(a \rightarrow ((b \rightarrow c) \rightarrow b)), \mu(a), \frac{1-k}{2}\}.$$

Then  $\mu(b) < s \leq \min\{\mu(a \rightarrow ((b \rightarrow c) \rightarrow b)), \mu(a), \frac{1-k}{2}\}$  for some  $s \in (0, \frac{1-k}{2}]$ . Thus  $(a \rightarrow ((b \rightarrow c) \rightarrow b))_s \in \mu$  and  $a_s \in \mu$ . Using Definition 3.3(2), we have  $b_s = b_{\min\{s, s\}} \in \vee q_k \mu$ , which implies that  $\mu(b) \geq s$  or  $\mu(b) + s + k > 1$ . This is a contradiction, and therefore

$$\mu(x) \geq \min\{\mu(a \rightarrow ((x \rightarrow y) \rightarrow x)), \mu(a), \frac{1-k}{2}\}$$

for all  $x, a, y \in L$ .

Conversely, let  $\mu$  be a fuzzy subset of  $L$  that satisfies two conditions (1) and (2). Let  $x \in L$  and  $t \in (0, 1]$  be such that  $x_t \in \mu$ . Then  $\mu(x) \geq t$ , which implies from (1) that  $\mu(1) \geq \min\{\mu(x), \frac{1-k}{2}\} \geq \min\{t, \frac{1-k}{2}\}$ . If  $t \leq \frac{1-k}{2}$ , then  $\mu(1) \geq t$ , i.e.,  $1_t \in \mu$ . If  $t > \frac{1-k}{2}$ , then  $\mu(1) \geq \frac{1-k}{2}$  and so  $\mu(1) + t + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$ , i.e.,  $1_t q_k \mu$ . Hence  $1_t \in \vee q_k \mu$ . Let  $x, a, y \in L$  and  $t, s \in (0, 1]$  be such that  $(a \rightarrow ((x \rightarrow y) \rightarrow x))_t \in \mu$  and  $a_s \in \mu$ . Then  $\mu(a \rightarrow ((x \rightarrow y) \rightarrow x)) \geq t$  and  $\mu(a) \geq s$ . It follows from (2) that

$$\begin{aligned} \mu(x) &\geq \min\{\mu(a \rightarrow ((x \rightarrow y) \rightarrow x)), \mu(a), \frac{1-k}{2}\} \\ &\geq \min\{t, s, \frac{1-k}{2}\}. \end{aligned}$$

If  $\min\{t, s\} \leq \frac{1-k}{2}$ , then  $\mu(x) \geq \min\{t, s\}$ , which shows that  $x_{\min\{t, s\}} \in \mu$ . If  $\min\{t, s\} > \frac{1-k}{2}$ , then  $\mu(x) \geq \frac{1-k}{2}$ , and thus  $\mu(x) + \min\{t, s\} + k > 1$ , i.e.,  $x_{\min\{t, s\}} q_k \mu$ . Hence  $x_{\min\{t, s\}} \in \vee q_k \mu$ . Consequently,  $\mu$  is an  $(\in, \in \vee q_k)$ -fuzzy fated filter of  $L$ . ■

**Corollary 3.9.** *If  $\mu$  is an  $(\in, \in \vee q_k)$ -fuzzy fated filter of  $L$  with  $\mu(1) < \frac{1-k}{2}$ , then  $\mu$  is an  $(\in, \in)$ -fuzzy fated filter of  $L$ .*

**Corollary 3.10.** *[[3]] A fuzzy subset  $\mu$  of  $L$  is an  $(\in, \in \vee q)$ -fuzzy fated filter of  $L$  if and only if it satisfies the following inequalities:*

- (1)  $(\forall x \in L) (\mu(1) \geq \min\{\mu(x), 0.5\})$ ,
- (2)  $(\forall x, a, y \in L) (\mu(x) \geq \min\{\mu(a \rightarrow ((x \rightarrow y) \rightarrow x)), \mu(a), 0.5\})$ .

*Proof.* It follows from taking  $k = 0$  in Theorem 3.8. ■

**Corollary 3.11.** *Every strong  $(\in, \in \vee q_k)$ -fuzzy fated filter  $\mu$  of  $L$  satisfies the following inequalities:*

- (1)  $(\forall x \in L) (\mu(1) \geq \min\{\mu(x), \frac{1-k}{2}\})$ ,
- (2)  $(\forall x, a, y \in L) (\mu(x) \geq \min\{\mu(a \rightarrow ((x \rightarrow y) \rightarrow x)), \mu(a), \frac{1-k}{2}\})$ .

**Theorem 3.12.** *A fuzzy subset  $\mu$  of  $L$  is an  $(\in, \in \vee q_k)$ -fuzzy fated filter of  $L$  if and only if it satisfies the following assertion:*

$$(\forall t \in (0, \frac{1-k}{2}]) (U(\mu; t) \in FF(L) \cup \{\emptyset\}). \quad (4)$$

*Proof.* Assume that  $\mu$  is an  $(\in, \in \vee q_k)$ -fuzzy fated filter of  $L$ . Let  $t \in (0, \frac{1-k}{2}]$  be such that  $U(\mu; t) \neq \emptyset$ . Then there exists  $x \in U(\mu; t)$ , and so  $\mu(x) \geq t$ . Using Theorem 3.8(1), we get

$$\mu(1) \geq \min\{\mu(x), \frac{1-k}{2}\} \geq \min\{t, \frac{1-k}{2}\} = t,$$

i.e.,  $1 \in U(\mu; t)$ . Assume that  $a \rightarrow ((x \rightarrow y) \rightarrow x) \in U(\mu; t)$  for all  $x, y \in L$  and  $a \in U(\mu; t)$ . Then  $\mu(a \rightarrow ((x \rightarrow y) \rightarrow x)) \geq t$  and  $\mu(a) \geq t$ . It follows from Theorem 3.8(2) that

$$\begin{aligned} \mu(x) &\geq \min \left\{ \mu(a \rightarrow ((x \rightarrow y) \rightarrow x)), \mu(a), \frac{1-k}{2} \right\} \\ &\geq \min \left\{ t, \frac{1-k}{2} \right\} = t \end{aligned}$$

so that  $x \in U(\mu; t)$ . Therefore  $U(\mu; t)$  is a fated filter of  $L$ .

Conversely, let  $\mu$  be a fuzzy subset of  $L$  satisfying the assertion (4). Assume that  $\mu(1) < \min \left\{ \mu(a), \frac{1-k}{2} \right\}$  for some  $a \in L$ . Putting  $t_a := \min \left\{ \mu(a), \frac{1-k}{2} \right\}$ , we have  $a \in U(\mu; t_a)$  and so  $U(\mu; t_a) \neq \emptyset$ . Hence  $U(\mu; t_a)$  is a fated filter of  $L$  by (4), which implies that  $1 \in U(\mu; t_a)$ . Thus  $\mu(1) \geq t_a$ , which is a contradiction. Therefore  $\mu(1) \geq \min \left\{ \mu(x), \frac{1-k}{2} \right\}$  for all  $x \in L$ . Suppose that

$$\mu(x) < \min \left\{ \mu(a \rightarrow ((x \rightarrow y) \rightarrow x)), \mu(a), \frac{1-k}{2} \right\}$$

for some  $x, a, y \in L$ . Taking  $t_x := \min \left\{ \mu(a \rightarrow ((x \rightarrow y) \rightarrow x)), \mu(a), \frac{1-k}{2} \right\}$ , we get  $a \rightarrow ((x \rightarrow y) \rightarrow x) \in U(\mu; t_x)$  and  $a \in U(\mu; t_x)$ . It follows from (1) that  $x \in U(\mu; t_x)$ , i.e.,  $\mu(x) \geq t_x$ . This is a contradiction. Hence

$$\mu(x) \geq \min \left\{ \mu(a \rightarrow ((x \rightarrow y) \rightarrow x)), \mu(a), \frac{1-k}{2} \right\}$$

for all  $x, a, y \in L$ . Using Theorem 3.8, we conclude that  $\mu$  is an  $(\in, \in \vee q_k)$ -fuzzy fated filter of  $L$ . ■

If we take  $k = 0$  in Theorem 3.12, then we have the following corollary.

**Corollary 3.13.** *[[3]] A fuzzy subset  $\mu$  of  $L$  is an  $(\in, \in \vee q)$ -fuzzy fated filter of  $L$  if and only if it satisfies the following assertion:*

$$(\forall t \in (0, 0.5]) \quad (U(\mu; t) \in FF(L) \cup \{\emptyset\}). \quad (5)$$

**Theorem 3.14.** *If  $k < r$  in  $[0, 1)$ , then every  $(\in, \in \vee q_k)$ -fuzzy fuzzy fated filter of  $L$  is an  $(\in, \in \vee q_r)$ -fuzzy fuzzy fated filter.*

*Proof.* Straightforward. ■

The converse of Theorem 3.14 is not true as seen in the following example.

**Example 3.15.** Consider an  $R_0$ -algebra  $L = \{0, a, b, c, d, 1\}$  which is appeared in Example 3.4. Define a fuzzy subset  $\mu$  of  $L$  by

$$\mu : L \rightarrow [0, 1], \quad x \mapsto \begin{cases} 0.9 & \text{if } x = d, \\ 0.7 & \text{if } x = c, \\ 0.3 & \text{if } x = 1, \\ 0.1 & \text{if } x \in \{0, a, b\}. \end{cases}$$



It is routine to verify that  $\mu$  is an  $(\in, \in \vee q_{0.4})$ -fuzzy fated filter of  $L$ . Since

$$U(\mu; t) = \begin{cases} \{c, d\} & \text{if } t \in (0.3, 0.35], \\ \{c, d, 1\} & \text{if } t \in (0.1, 0.3], \\ L & \text{if } t \in (0, 0.1], \end{cases}$$

we know from Theorem 3.12 that  $\mu$  is not an  $(\in, \in \vee q_{0.3})$ -fuzzy fated filter of  $L$ .

**Proposition 3.16.** *Every  $(\in, \in \vee q_k)$ -fuzzy fated filter  $\mu$  of  $L$  satisfies the following inequalities.*

- (1)  $\mu(x \rightarrow z) \geq \min \left\{ \mu(x \rightarrow (y \rightarrow z)), \mu(x \rightarrow y), \frac{1-k}{2} \right\},$
- (2)  $\mu(x) \geq \min \left\{ \mu((x \rightarrow y) \rightarrow x), \frac{1-k}{2} \right\}$

for all  $x, y, z \in L$ .

*Proof.* (1) Suppose that there exist  $a, b, c \in L$  such that

$$\mu(a \rightarrow c) < \min \left\{ \mu(a \rightarrow (b \rightarrow c)), \mu(a \rightarrow b), \frac{1-k}{2} \right\}.$$

Taking  $t := \min \left\{ \mu(a \rightarrow (b \rightarrow c)), \mu(a \rightarrow b), \frac{1-k}{2} \right\}$  implies that  $a \rightarrow (b \rightarrow c) \in U(\mu; t)$ ,  $a \rightarrow b \in U(\mu; t)$  and  $t \in (0, \frac{1-k}{2}]$ . Since  $U(\mu; t) \in FF(L)$  by Theorem 3.12, it follows from Lemma 3.1 that  $a \rightarrow c \in U(\mu; t)$ , i.e.,  $\mu(a \rightarrow c) \geq t$ . This is a contradiction, and therefore  $\mu$  satisfies (1).

(2) If  $\mu$  is an  $(\in, \in \vee q_k)$ -fuzzy fated filter of  $L$ , then  $U(\mu; t) \in FF(L) \cup \{\emptyset\}$  for all  $t \in (0, \frac{1-k}{2}]$  by Theorem 3.12. Hence  $U(\mu; t) \in F(L) \cup \{\emptyset\}$  for all  $t \in (0, \frac{1-k}{2}]$ . Suppose that

$$\mu(x) < t \leq \min \left\{ \mu((x \rightarrow y) \rightarrow x), \frac{1-k}{2} \right\}$$

for some  $x, y \in L$  and  $t \in (0, \frac{1-k}{2}]$ . Then  $(x \rightarrow y) \rightarrow x \in U(\mu; t)$ , which implies from Lemma 3.2 that  $x \in U(\mu; t)$ , i.e.,  $\mu(x) \geq t$ . This is a contradiction. Hence  $\mu(x) \geq \min \left\{ \mu((x \rightarrow y) \rightarrow x), \frac{1-k}{2} \right\}$  for all  $x, y \in L$ . ■

**Corollary 3.17.** *[[3]] Every  $(\in, \in \vee q)$ -fuzzy fated filter  $\mu$  of  $L$  satisfies the following inequalities.*

- (1)  $\mu(x \rightarrow z) \geq \min \{ \mu(x \rightarrow (y \rightarrow z)), \mu(x \rightarrow y), 0.5 \},$
- (2)  $\mu(x) \geq \min \{ \mu((x \rightarrow y) \rightarrow x), 0.5 \}$

for all  $x, y, z \in L$ .

**Theorem 3.18.** *If  $F$  is a fated filter of  $L$ , then a fuzzy subset  $\mu$  of  $L$  defined by*

$$\mu : L \rightarrow [0, 1], \quad x \mapsto \begin{cases} t_1 & \text{if } x \in F, \\ t_2 & \text{otherwise} \end{cases}$$

where  $t_1 \in [\frac{1-k}{2}, 1]$  and  $t_2 \in (0, \frac{1-k}{2})$ , is an  $(\in, \in \vee q_k)$ -fuzzy fated filter of  $L$ .

*Proof.* Note that

$$U(\mu; s) = \begin{cases} F & \text{if } s \in (t_2, \frac{1-k}{2}], \\ L & \text{if } s \in (0, t_2] \end{cases}$$

which is a fated filter of  $L$ . It follows from Theorem 3.12 that  $\mu$  is an  $(\in, \in \vee q_k)$ -fuzzy fated filter of  $L$ . ■

**Corollary 3.19.** *[[3]] If  $F$  is a fated filter of  $L$ , then a fuzzy subset  $\mu$  of  $L$  defined by*

$$\mu : L \rightarrow [0, 1], \quad x \mapsto \begin{cases} t_1 & \text{if } x \in F, \\ t_2 & \text{otherwise} \end{cases}$$

where  $t_1 \in [0.5, 1]$  and  $t_2 \in (0, 0.5)$ , is an  $(\in, \in \vee q)$ -fuzzy fated filter of  $L$ .

For any fuzzy subset  $\mu$  of  $L$  and any  $t \in (0, 1]$ , we consider two subsets:

$$Q(\mu; t) := \{x \in L \mid x_t q \mu\}, \quad [\mu]_t := \{x \in L \mid x_t \in \vee q \mu\}.$$

It is clear that  $[\mu]_t = U(\mu; t) \cup Q(\mu; t)$  (see [3]). We also consider the following two sets:

$$Q_k(\mu; t) := \{x \in L \mid x_t q_k \mu\}, \quad [\mu]_t^k := \{x \in L \mid x_t \in \vee q_k \mu\}.$$

Obviously,  $[\mu]_t^k = U(\mu; t) \cup Q_k(\mu; t)$  and if  $k = 0$  then  $Q_k(\mu; t) = Q(\mu; t)$  and  $[\mu]_t^k = [\mu]_t$ .

**Theorem 3.20.** *If  $\mu$  is an  $(\in, \in \vee q_k)$ -fuzzy fated filter of  $L$ , then*

$$(\forall t \in (\frac{1-k}{2}, 1]) \quad (Q_k(\mu; t) \in FF(L) \cup \{\emptyset\}).$$

*Proof.* Assume that  $\mu$  is an  $(\in, \in \vee q_k)$ -fuzzy fated filter of  $L$  and let  $t \in (\frac{1-k}{2}, 1]$  be such that  $Q_k(\mu; t) \neq \emptyset$ . Then there exists  $x \in Q_k(\mu; t)$ , and so  $\mu(x) + t + k > 1$ . Using Theorem 3.8(1), we have

$$\begin{aligned} \mu(1) &\geq \min \left\{ \mu(x), \frac{1-k}{2} \right\} \\ &= \begin{cases} \frac{1-k}{2} & \text{if } \mu(x) \geq \frac{1-k}{2}, \\ \mu(x) & \text{if } \mu(x) < \frac{1-k}{2} \end{cases} \\ &> 1 - t - k, \end{aligned}$$

which implies that  $1 \in Q_k(\mu; t)$ . Assume that  $a \rightarrow ((x \rightarrow y) \rightarrow x) \in Q_k(\mu; t)$  and  $a \in Q_k(\mu; t)$  for all  $x, a, y \in L$ . Then  $(a \rightarrow ((x \rightarrow y) \rightarrow x))_t q_k \mu$  and  $a_t q_k \mu$ , that is,  $\mu(a \rightarrow ((x \rightarrow y) \rightarrow x)) > 1 - t - k$  and  $\mu(a) > 1 - t - k$ . Using Theorem 3.8(2), we get

$$\mu(x) \geq \min \left\{ \mu(a \rightarrow ((x \rightarrow y) \rightarrow x)), \mu(a), \frac{1-k}{2} \right\}.$$

Thus, if  $\min\{\mu(a \rightarrow ((x \rightarrow y) \rightarrow x)), \mu(a)\} < \frac{1-k}{2}$ , then

$$\mu(x) \geq \min\{\mu(a \rightarrow ((x \rightarrow y) \rightarrow x)), \mu(a)\} > 1 - t - k.$$

If  $\min\{\mu(a \rightarrow ((x \rightarrow y) \rightarrow x)), \mu(a)\} \geq \frac{1-k}{2}$ , then  $\mu(x) \geq \frac{1-k}{2} > 1 - t - k$ . It follows that  $x_t q_k \mu$  so that  $x \in Q_k(\mu; t)$ . Therefore  $Q_k(\mu; t)$  is a fated filter of  $L$ .  $\blacksquare$

**Corollary 3.21.** *[[3]] If  $\mu$  is an  $(\in, \in \vee q)$ -fuzzy fated filter of  $L$ , then*

$$(\forall t \in (0.5, 1]) \quad (Q(\mu; t) \in FF(L) \cup \{\emptyset\}).$$

**Corollary 3.22.** *If  $\mu$  is a strong  $(\in, \in \vee q_k)$ -fuzzy fated filter of  $L$ , then*

$$(\forall t \in (\frac{1-k}{2}, 1]) \quad (Q_k(\mu; t) \in FF(L) \cup \{\emptyset\}).$$

The converse of Corollary 3.22 is not true as shown by the following example.

*Example 3.23.* Consider the  $(\in, \in \vee q_{0.2})$ -fuzzy fated filter  $\nu$  of  $L$  which is given in Example 3.4. Then

$$Q_k(\nu; t) = \begin{cases} L & \text{if } t \in (0.5, 1], \\ \{c, d, 1\} & \text{if } t \in (0.4, 0.5] \end{cases}$$

is a fated filter of  $L$ . But  $\nu$  is not a strong  $(\in, \in \vee q_{0.2})$ -fuzzy fated filter of  $L$ .

**Theorem 3.24.** *For a fuzzy subset  $\mu$  of  $L$ , the following assertions are equivalent:*

- (1)  $\mu$  is an  $(\in, \in \vee q_k)$ -fuzzy fated filter of  $L$ .
- (2)  $(\forall t \in (0, 1]) \quad ([\mu]_t^k \in FF(L) \cup \{\emptyset\})$ .

We call  $[\mu]_t^k$  an  $(\in \vee q_k)$ -level fated filter of  $\mu$ .

*Proof.* Assume that  $\mu$  is an  $(\in, \in \vee q_k)$ -fuzzy fated filter of  $L$  and let  $t \in (0, 1]$  be such that  $[\mu]_t^k \neq \emptyset$ . Then there exists  $x \in [\mu]_t^k = U(\mu; t) \cup Q_k(\mu; t)$ , and so  $x \in U(\mu; t)$  or  $x \in Q_k(\mu; t)$ . If  $x \in U(\mu; t)$ , then  $\mu(x) \geq t$ . It follows from Theorem 3.8(1) that

$$\begin{aligned} \mu(1) &\geq \min\{\mu(x), \frac{1-k}{2}\} \geq \min\{t, \frac{1-k}{2}\} \\ &= \begin{cases} t & \text{if } t \leq \frac{1-k}{2}, \\ \frac{1-k}{2} > 1 - t - k & \text{if } t > \frac{1-k}{2} \end{cases} \end{aligned}$$

so that  $1 \in U(\mu; t) \cup Q_k(\mu; t) = [\mu]_t^k$ . If  $x \in Q_k(\mu; t)$ , then  $\mu(x) + t + k > 1$ . Thus

$$\begin{aligned} \mu(1) &\geq \min\{\mu(x), \frac{1-k}{2}\} \geq \min\{1 - t - k, \frac{1-k}{2}\} \\ &= \begin{cases} 1 - t - k & \text{if } t > \frac{1-k}{2}, \\ \frac{1-k}{2} \geq t & \text{if } t \leq \frac{1-k}{2} \end{cases} \end{aligned}$$

and so  $1 \in Q_k(\mu; t) \cup U(\mu; t) = [\mu]_t^k$ . Let  $x, a, y \in L$  be such that  $a \in [\mu]_t^k$  and  $a \rightarrow ((x \rightarrow y) \rightarrow x) \in [\mu]_t^k$ . Then

$$\mu(a) \geq t \text{ or } \mu(a) + t + k > 1,$$

and

$$\mu(a \rightarrow ((x \rightarrow y) \rightarrow x)) \geq t \text{ or } \mu(a \rightarrow ((x \rightarrow y) \rightarrow x)) + t + k > 1.$$

We can consider four cases:

$$\mu(a) \geq t \text{ and } \mu(a \rightarrow ((x \rightarrow y) \rightarrow x)) \geq t, \quad (6)$$

$$\mu(a) \geq t \text{ and } \mu(a \rightarrow ((x \rightarrow y) \rightarrow x)) + t + k > 1, \quad (7)$$

$$\mu(a) + t + k > 1 \text{ and } \mu(a \rightarrow ((x \rightarrow y) \rightarrow x)) \geq t, \quad (8)$$

$$\mu(a) + t + k > 1 \text{ and } \mu(a \rightarrow ((x \rightarrow y) \rightarrow x)) + t + k > 1. \quad (9)$$

For the first case, Theorem 3.8(2) implies that

$$\begin{aligned} \mu(x) &\geq \min\{\mu(a \rightarrow ((x \rightarrow y) \rightarrow x)), \mu(a), \frac{1-k}{2}\} \\ &\geq \min\{t, \frac{1-k}{2}\} = \begin{cases} \frac{1-k}{2} & \text{if } t > \frac{1-k}{2}, \\ t & \text{if } t \leq \frac{1-k}{2} \end{cases} \end{aligned}$$

so that  $x \in U(\mu; t)$  or  $\mu(x) + t + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$ , i.e.,  $x \in Q_k(\mu; t)$ . Hence  $x \in [\mu]_t^k$ . Case (7) implies that

$$\begin{aligned} \mu(x) &\geq \min\{\mu(a \rightarrow ((x \rightarrow y) \rightarrow x)), \mu(a), \frac{1-k}{2}\} \\ &\geq \min\{1 - t - k, t, \frac{1-k}{2}\} = \begin{cases} 1 - t - k & \text{if } t > \frac{1-k}{2}, \\ t & \text{if } t \leq \frac{1-k}{2}. \end{cases} \end{aligned}$$

Thus  $x \in Q_k(\mu; t) \cup U(\mu; t) = [\mu]_t^k$ . Similarly,  $x \in [\mu]_t^k$  for the case (8). The final case implies that

$$\begin{aligned} \mu(x) &\geq \min\{\mu(a \rightarrow ((x \rightarrow y) \rightarrow x)), \mu(a), \frac{1-k}{2}\} \\ &\geq \min\{1 - t - k, \frac{1-k}{2}\} = \begin{cases} 1 - t - k & \text{if } t > \frac{1-k}{2}, \\ \frac{1-k}{2} & \text{if } t \leq \frac{1-k}{2} \end{cases} \end{aligned}$$

so that  $x \in Q_k(\mu; t) \cup U(\mu; t) = [\mu]_t^k$ . Consequently  $[\mu]_t^k$  is a fuzzy fated filter of  $L$ .

Conversely, let  $\mu$  be a fuzzy subset of  $L$  such that  $[\mu]_t^k$  is a fated filter of  $L$  whenever it is nonempty for all  $t \in (0, 1]$ . If there exists  $a \in L$  such that  $\mu(1) < \min\{\mu(a), \frac{1-k}{2}\}$ , then  $\mu(1) < t_a \leq \min\{\mu(a), \frac{1-k}{2}\}$  for some  $t_a \in (0, \frac{1-k}{2}]$ . It follows that  $a \in U(\mu; t_a)$  but  $1 \notin U(\mu; t_a)$ . Also,  $\mu(1) + t_a + k < 2t_a + k \leq 1$  and so  $1 \notin Q_k(\mu; t_a)$ . Hence  $1 \notin U(\mu; t_a) \cup Q_k(\mu; t_a) = [\mu]_{t_a}^k$ , which is a contradiction. Therefore  $\mu(1) \geq \min\{\mu(x), \frac{1-k}{2}\}$  for all  $x \in L$ . Suppose that

$$\mu(x) < \min\{\mu(a \rightarrow ((x \rightarrow y) \rightarrow x)), \mu(a), \frac{1-k}{2}\} \quad (10)$$

for some  $x, a, y \in L$ . Taking  $t := \min\{\mu(a \rightarrow ((x \rightarrow y) \rightarrow x)), \mu(a), \frac{1-k}{2}\}$  implies that  $t \in (0, \frac{1-k}{2}]$ ,  $a \in U(\mu; t) \subseteq [\mu]_t^k$  and  $a \rightarrow ((x \rightarrow y) \rightarrow x) \in U(\mu; t) \subseteq [\mu]_t^k$ . Since  $[\mu]_t^k \in FF(L)$ , it follows that  $x \in [\mu]_t^k = U(\mu; t) \cup Q_k(\mu; t)$ . But (10) induces  $x \notin U(\mu; t)$  and  $\mu(x) + t + k < 2t + k \leq 1$ , i.e.,  $x \notin Q_k(\mu; t)$ . This is a contradiction, and thus  $\mu(x) \geq \min\{\mu(a \rightarrow ((x \rightarrow y) \rightarrow x)), \mu(a), \frac{1-k}{2}\}$  for all  $x, a, y \in L$ . Using Theorem 3.8, we conclude that  $\mu$  is an  $(\in, \in \vee q_k)$ -fuzzy fated filter of  $L$ . ■

**Corollary 3.25.** *[[3]] For a fuzzy subset  $\mu$  of  $L$ , the following assertions are equivalent:*

- (1)  $\mu$  is an  $(\in, \in \vee q)$ -fuzzy fated filter of  $L$ .
- (2)  $(\forall t \in (0, 1]) ([\mu]_t \in FF(L) \cup \{\emptyset\})$ .

*Proof.* Taking  $k = 0$  in Theorem 3.24 induced the desired result. ■

**Theorem 3.26.** *Given any chain of fated filters  $F_0 \subset F_1 \subset \dots \subset F_n = L$  of  $L$ , there exists an  $(\in, \in \vee q_k)$ -fuzzy fated filter  $\mu$  of  $L$  whose level fated filters are precisely the members of the chain with  $U(\mu; \frac{1-k}{2}) = F_0$ .*

*Proof.* Let  $\{t_i \in (0, \frac{1-k}{2}) \mid i = 1, 2, \dots, n\}$  be such that  $t_1 > t_2 > \dots > t_n$ . Define a fuzzy subset  $\mu$  of  $L$  by

$$\mu : L \rightarrow [0, 1], \quad x \mapsto \begin{cases} t_0 (\geq \frac{1-k}{2}) & \text{if } x = 1, \\ t (\geq t_0) & \text{if } x \in F_0 \setminus \{1\}, \\ t_1 & \text{if } x \in F_1 \setminus F_0, \\ t_2 & \text{if } x \in F_2 \setminus F_1, \\ \dots & \\ t_n & \text{if } x \in F_n \setminus F_{n-1}. \end{cases}$$

Then

$$U(\mu; s) = \begin{cases} F_0 & \text{if } s \in (t_1, \frac{1-k}{2}], \\ F_1 & \text{if } s \in (t_2, t_1], \\ F_2 & \text{if } s \in (t_3, t_2], \\ \dots & \\ F_n = R & \text{if } s \in (0, t_n]. \end{cases}$$

Using Theorem 3.12, we know that  $\mu$  is an  $(\in, \in \vee q_k)$ -fuzzy fated filter of  $L$ . It follows from the construction of  $\mu$  that  $U(\mu; \frac{1-k}{2}) = F_0$  and  $U(\mu; t_i) = F_i$  for  $i = 1, 2, \dots, n$ . ■

**Corollary 3.27.** *Given any chain of fated filters  $F_0 \subset F_1 \subset \dots \subset F_n = L$  of  $L$ , there exists an  $(\in, \in \vee q)$ -fuzzy fated filter  $\mu$  of  $L$  whose level fated filters are precisely the members of the chain with  $U(\mu; 0.5) = F_0$ .*

Using a class of fated filters, we make an  $(\in, \in \vee q_k)$ -fuzzy fated filter of  $L$ .

**Theorem 3.28.** *Let  $\{F_t \mid t \in \Lambda\}$ , where  $\Lambda \subseteq (0, \frac{1-k}{2}]$ , be a collection of fated filters of  $L$  such that*

- (i)  $L = \bigcup_{t \in \Lambda} F_t$ ,
- (ii)  $(\forall s, t \in \Lambda) (s < t \Leftrightarrow F_t \subset F_s)$ .

*Then a fuzzy subset  $\mu$  of  $L$  defined by  $\mu(x) = \sup\{t \in \Lambda \mid x \in F_t\}$  for all  $x \in L$  is an  $(\in, \in \vee q_k)$ -fuzzy fated filter of  $L$ .*

*Proof.* According to Theorem 3.12, it is sufficient to show that  $U(\mu; t) \neq \emptyset$  is a fated filter of  $L$  for all  $t \in (0, \frac{1-k}{2}]$ . We consider two cases:

- (i)  $t = \sup\{s \in \Lambda \mid s < t\}$ ,
- (ii)  $t \neq \sup\{s \in \Lambda \mid s < t\}$ .

Case (i) implies that

$$x \in U(\mu; t) \iff (x \in F_s, \forall s < t) \iff x \in \bigcap_{s < t} F_s,$$

and so  $U(\mu; t) = \bigcap_{s < t} F_s$  which is a fated filter of  $L$ . In the second case, we have  $U(\mu; t) = \bigcup_{s \geq t} F_s$ . Indeed, if  $x \in \bigcup_{s \geq t} F_s$ , then  $x \in F_s$  for some  $s \geq t$ . Thus  $\mu(x) \geq s \geq t$ , i.e.,  $x \in U(\mu; t)$ . This proves  $\bigcup_{s \geq t} F_s \subset U(\mu; t)$ . To prove the reverse inclusion, let  $x \notin \bigcup_{s \geq t} F_s$ . Then  $x \notin F_s$  for all  $s \geq t$ . Since  $t \neq \sup\{s \in \Lambda \mid s < t\}$ , there exists  $\varepsilon > 0$  such that  $(t - \varepsilon, t) \cap \Lambda = \emptyset$ . Hence  $x \notin F_s$  for all  $s > t - \varepsilon$ , which means that if  $x \in F_s$  then  $s \leq t - \varepsilon$ . Thus  $\mu(x) \leq t - \varepsilon < t$ , and so  $x \notin U(\mu; t)$ . Therefore  $U(\mu; t) = \bigcup_{s \geq t} F_s$  which is also a fated filter of  $L$ . Consequently,  $\mu$  is an  $(\in, \in \vee q_k)$ -fuzzy fated filter of  $L$ . ■

**Corollary 3.29.** *Let  $\{F_t \mid t \in \Lambda\}$ , where  $\Lambda \subseteq (0, 0.5]$ , be a collection of fated filters of  $L$  such that*

- (i)  $L = \bigcup_{t \in \Lambda} F_t$ ,
- (ii)  $(\forall s, t \in \Lambda) (s < t \Leftrightarrow F_t \subset F_s)$ .

*Then a fuzzy subset  $\mu$  of  $L$  defined by  $\mu(x) = \sup\{t \in \Lambda \mid x \in F_t\}$  for all  $x \in L$  is an  $(\in, \in \vee q)$ -fuzzy fated filter of  $L$ .*

A fuzzy subset  $\mu$  of  $L$  is said to be *proper* if  $\text{Im}(\mu)$  has at least two elements. Two fuzzy subsets are said to be *equivalent* if they have same family of level subsets. Otherwise, they are said to be *non-equivalent*.

**Theorem 3.30.** *Let  $\mu$  be an  $(\in, \in \vee q_k)$ -fuzzy fated filter of  $L$  such that  $\#\{\mu(x) \mid \mu(x) < \frac{1-k}{2}\} \geq 2$ . Then there exist two proper non-equivalent  $(\in, \in \vee q_k)$ -fuzzy fated filters of  $L$  such that  $\mu$  can be expressed as the union of them.*

*Proof.* Let  $\{\mu(x) \mid \mu(x) < \frac{1-k}{2}\} = \{t_1, t_2, \dots, t_r\}$ , where  $t_1 > t_2 > \dots > t_r$  and  $r \geq 2$ . Then the chain of  $(\in \vee q_k)$ -level fated filters of  $\mu$  is

$$[\mu]_{\frac{1-k}{2}}^k \subseteq [\mu]_{t_1}^k \subseteq [\mu]_{t_2}^k \subseteq \dots \subseteq [\mu]_{t_r}^k = L.$$

Let  $\nu$  and  $\gamma$  be fuzzy subsets of  $L$  defined by

$$\nu(x) = \begin{cases} t_1 & \text{if } x \in [\mu]_{t_1}^k, \\ t_2 & \text{if } x \in [\mu]_{t_2}^k \setminus [\mu]_{t_1}^k, \\ \dots & \\ t_r & \text{if } x \in [\mu]_{t_r}^k \setminus [\mu]_{t_{r-1}}^k, \end{cases}$$

and

$$\gamma(x) = \begin{cases} \mu(x) & \text{if } x \in [\mu]_{\frac{1-k}{2}}^k, \\ k & \text{if } x \in [\mu]_{t_2}^k \setminus [\mu]_{\frac{1-k}{2}}^k, \\ t_3 & \text{if } x \in [\mu]_{t_3}^k \setminus [\mu]_{t_2}^k, \\ \dots & \\ t_r & \text{if } x \in [\mu]_{t_r}^k \setminus [\mu]_{t_{r-1}}^k, \end{cases}$$

respectively, where  $t_3 < k < t_2$ . Then  $\nu$  and  $\gamma$  are  $(\in, \in \vee q_k)$ -fuzzy fated filters of  $L$ , and  $\nu, \gamma \leq \mu$ . The chains of  $(\in \vee q_k)$ -level fated filters of  $\nu$  and  $\gamma$  are, respectively, given by

$$[\mu]_{t_1}^k \subseteq [\mu]_{t_2}^k \subseteq \dots \subseteq [\mu]_{t_r}^k$$

and

$$[\mu]_{\frac{1-k}{2}}^k \subseteq [\mu]_{t_2}^k \subseteq \dots \subseteq [\mu]_{t_r}^k.$$

Therefore  $\nu$  and  $\gamma$  are non-equivalent and clearly  $\mu = \nu \cup \gamma$ . This completes the proof.  $\blacksquare$

#### 4. Conclusion

In this paper, using the "belongs to" relation  $(\in)$  and quasicoincidence with relation  $(q)$  between the fuzzy point and fuzzy sets, we introduce the notions of  $(\in, \in \vee q_k)$ -fuzzy fated filter in an  $R_0$ -algebras and investigate some related properties. We have dealt with characterizations of an  $(\in, \in \vee q_k)$ -fuzzy fated filter in  $R_0$ -algebras and have obtained an  $(\in, \in \vee q_k)$ -fuzzy fated filter is that the notion of an  $(\in, \in \vee q)$ -fuzzy fated filter is a special case of an  $(\in, \in \vee q_k)$ -fuzzy fated filter. Based on these results, we shall focus on other types and their relationships among them, and also consider these generalized rough fuzzy filters of  $R_0$ -algebras.

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